Mathematical Proofs

First: logic!

A logical statement: a sentence that is

1) declarative (i.e. not a question)

2) either true or false, but not both.

Example 1: (statements and not...)

1) "I am the chosen one."

2) "A zebra can not change its spots."

3) "When the president does it, that means it's not illegal."

4) "Every two child did."

1) This is a statement, but very hard to prove, depending on how you define "chosen".

2) Declarative sentence that is vacuously true. It is a statement that doesn't apply to anything!

3) This is a statement that is seemingly false...

4) Nonsense. Not a statement since it cannot even be logically parsed as given.

Universally and Existentially Quantified Statements

A "universally quantified" statement is a statement that applies for all objects in a certain class. For example, "All babies have blue eyes".

An "existentially quantified" statement is a statement that asserts there is an object with certain properties. For example, "There is a baby with brown eyes".

Negating Quantified Statements

Negation in logic = applying the word "not" to a statement.

The negation of a universally quantified statement is an existentially quantified statement. Similarly, the negation of an existentially quantified statement is a universally quantified statement.

Example 2: (negations)

1) Negate "All babies have blue eyes":

"Not all babies have blue eyes"

same as

"There is a baby that doesn't have blue eyes".

2) Negate "There is a car parked in my driveway":

"There is not a car parked in my driveway"

same as

"For all cars, they are not in my driveway".

Mathematical Proofs

Universal vs. Existential

To prove something universally quantified, you need to prove for all given objects. This usually involves variables/abstraction.

To prove something existentially quantified, you just need to show one example. This usually involves concrete, explicit numbers.

Example 3: Prove that every vector in $\mathbb{R}^2$ is a linear combination of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

Solution: This wants us to show all vectors in $\mathbb{R}^2$ are linear combinations of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$. This is a universal quantifier, so use abstraction.

A generic vector in $\mathbb{R}^2$ looks like $\begin{bmatrix} x \\ y \end{bmatrix}$ with x, y real numbers.

We want to show: there are constants c_1, c_2 with

$$c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Cheat: matrix-vector multiplication!

$$c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 + 4c_2 \\ -c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ \begin{bmatrix} -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ \begin{bmatrix} -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

This quantity is supposed to equal

$\begin{bmatrix} x \\ y \end{bmatrix}$, so :

$$\begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{RREF} \left[\begin{array}{cc|c} 1 & 4 & x \\ -1 & 2 & y \end{array}\right]$$

We get

$$\begin{array}{ccc} c_1 & c_2 & \text{Solutions} \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 0 & \frac{1}{3}(x-2y) \\ 0 & 1 & \frac{1}{6}(x+y) \end{array}\right]$$

So $c_1 = \frac{1}{3}(x-2y)$, $c_2 = \frac{1}{6}(x+y)$

and there are always solutions!

So every vector in $\mathbb{R}^2$ is a linear combination of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

Example 4: Show that not every vector in $\mathbb{R}^3$ is a linear combination of $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$.

Solution: Negating a universal quantifier = existential quantifier.

Find me <u>one</u> vector that is not a linear combination of these three!

For example, try $\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$

Not yet a proof

We want to show there do not exist constants c_1, c_2, and c_3 with

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$$

Following the same steps as in the previous example, we can go from this to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ -1 & 4 & 2 & 2 \\ 1 & 3 & 5 & -1 \end{array}\right]$$

put in RREF

We get

$$\begin{array}{cccc} c_1 & c_2 & c_3 & \text{solutions} \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Last row says no solution, so

$\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$ is not a linear

combination of

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}, \text{ and } \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}.$$

Conditional Statements

These are statements of the form:

"If it is raining, then I carry my umbrella".

"If - then" statements are called conditional statements.

Example 5: Consider the following conditional statements:

"If $\vec{x}$ is a vector in $\mathbb{R}^3$, then $\operatorname{span}(\{\vec{x}\})$ is a line."

"If the earth is flat, then I will eat my lunch outside today".

The first statement is technically false since $\vec{x}$ could be the zero vector.

The second statement is another example of a vacuously true statement. Since the earth is not flat, the condition will never be met, so we can conclude anything from it!

Negating a Conditional

Negate "If it is raining then I bring my umbrella".

It is "It is raining and I didn't bring my umbrella."

not a conditional statement!

In general, the negation of a conditional statement is not a conditional statement.

The Most Important Rule in Proofs:

Don't assume what you're trying to prove!